Time-decay of semigroups generated by dissipative Schrödinger operators

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TIME-DECAY OF THE SEMIGROUP OF DISSIPATIVE SCHRÖDINGER OPERATORS

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Abstract. We establish a representation formula for semigroup of contraction in terms of global limiting absorption principle from the upper-half complex plane. As applications, we prove time-decay estimates of the semigroup of contractions generated by $-iH$ where H is a dissipative Schrödinger operator.

1. Introduction

Let $H = -\Delta + V(x)$ be the Schrödinger operator with a complex-valued potential V satisfying $V = V_1 - iV_2$, where V_1 and V_2 are real functions satisfying $V_2(x) \geq 0$ and $V_2(x) > 0$ on some non-trivial open set. Suppose that

$$|V_j(x)| \leq C\langle x\rangle^{-\rho_0}, \quad x \in \mathbb{R}^n, \tag{1.1}$$

ass1

for some $\rho_0 > 1$. Here $\langle x\rangle = (1 + |x|^2)^{1/2}$. Mild local singularities can be included with little additional effort. Denote $H_0 = -\Delta$ and $H_1 = -\Delta + V_1$. H defined on $D(-\Delta)$ is maximally dissipative and the numerical range of H is contained in $\{z; \Re z \geq -R, -R \leq \Im z \leq 0\}$ for some $R > 0$.

2. Some abstract results

Let H_1 and V_2 be selfadjoint operators on some Hilbert space $\mathcal{H}$, with H_1 semi-bounded from below, $V_2 \geq 0$ and relatively compact with respect to H_1. $H = H_1 - iV_2$ is maximally dissipative on $\mathcal{H}$. Let $S(t) := e^{-itH}$, $t \geq 0$, be the strongly continuous semigroup generated by $-iH$. In this Section, we give two results on $S(t)$ based on the existence of a limiting absorption principle for H on the whole real axis. There results are to be applied in the next Section to a class of dissipative Schrödinger operators on $\mathbb{R}^n$, $n \geq 2$.

Since V_2 is H_1 compact, one sees that $\forall \delta_0 > 0$, $\exists R_1 > 0$ such that the numerical range of H is contained in the sector

$$\{z \in \mathbb{C}; \Re z > -R_1, -\delta_0 \leq \arg(z + R_1) \leq 0\}.$$

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Let $\delta_0 < \pi$ be fixed. For $\epsilon_0 > 0$ small enough and for all $\epsilon \in]0, \epsilon_0]$, the set $\{\lambda e^{i\epsilon}; \lambda \in \mathbb{R}, |\lambda| > R_1\}$ is contained in the resolvent set of H. Assume that there exists a dense subset $\mathcal{D} \subset \mathcal{H}$ such that $\{f \in \mathcal{D} \cap D(H); Hf \in \mathcal{D}\}$ is dense in $\mathcal{H}$ and

- For any $\lambda \in \mathbb{R}$, the limit
$$\langle R(\lambda + i0)f, g\rangle = \lim_{\epsilon\to 0_+} \langle R(\lambda + i\epsilon)f, g\rangle \tag{2.1}$$
exists for any $f, g \in \mathcal{D}$ and is continuous in $\lambda \in \mathbb{R}$.
- There exist $R > 1$, $k \in \mathbb{N}^*$, $\rho, \sigma > 0$ with $\rho + k\sigma > 1$ such that for any f, g in $\mathcal{D}$, $\lambda \to \langle R(\lambda + i0)f, g\rangle$ is C^k for $|\lambda| > R$ and
$$|\frac{d^j}{d\lambda^j}\langle R(\lambda + i0)f, g\rangle| \le C_{f,g}\langle\lambda\rangle^{-\rho-j\sigma}, \tag{2.2}$$
for $j = 0, 1, \ldots, k$ and $|\lambda| > R$.

Theorem 2.1. *Under the conditions (2.1) and (2.2), one has*
$$\langle e^{-itH}f, g\rangle = \frac{1}{2\pi i}\int_{\mathbb{R}} e^{-it\lambda}\langle R(\lambda + i0)f, g\rangle d\lambda, \quad t > 0, \tag{2.3}$$
for $f, g \in \mathcal{D}$.

Proof. Denote $H_\epsilon = e^{-i\epsilon}H$, $\epsilon > 0$. Then the numerical range of H_ϵ is contained in
$$\mathcal{N}_\epsilon := \{e^{-i\epsilon}z;\ \Re z > -R_1, -\delta_0 \le \arg(z + R_1) \le 0\}$$
for some $R_1 > 1$. Therefore for each $\epsilon > 0$ small enough, $H_\epsilon - iR_1\epsilon$ is maximally dissipative and strictly m-sectorial and $-iH_\epsilon$ generates a semigroup e^{-itH_ϵ}, $t \ge 0$, which can be represented in a usual way (cf. [12], pp 489-491). For $R_0 > \epsilon R_1$, let $\Gamma_{\epsilon,R}$ be a contour in $\rho(H_\epsilon)$ composed of the segment $\{\Im z = R_0, \Re z \in [-R_1 - 1, R_1 + 1]\}$ and the two rays $(-R_1 - 1 + iR_0) + e^{i(\pi+\epsilon/2)}\mathbb{R}_+$ and $(R_1 + 1 + iR_0) + e^{-i\epsilon/2}\mathbb{R}_+$ (running from the infinity with $\arg z = -\pi + \frac{\epsilon}{2}$ to the infinity with $\arg z = -\frac{\epsilon}{2}$). Then one has
$$e^{-itH_\epsilon} = \frac{1}{2\pi i}\int_{\Gamma_{\epsilon,R_0}} e^{-itz}(H_\epsilon - z)^{-1}dz := F_\epsilon(t), \tag{2.4}$$
for $t > 0$. In addition, one has the estimate
$$\|e^{-itH_\epsilon}\| \le e^{tR_1\epsilon}, \quad t \ge 0. \tag{2.5}$$

Under the condition (2.2), by an argument of perturbation, one can deduce that for $j = 0, 1, \cdots, k$,
$$|\frac{d^j}{d\lambda^j}\langle R(\lambda e^{i\eta})f, g\rangle| \le C_{f,g}\langle\lambda\rangle^{-\rho-j\sigma}, \tag{2.6}$$
uniformly in $\lambda \ge 1$ and $\eta > 0$ small enough. Making use of techniques of oscillatory integrals, we deduce from (2.1) and (2.2) that the integral
$$\langle F(t)f, g\rangle := \frac{1}{2\pi i}\int_{\mathbb{R}} e^{-it\lambda}\langle R(\lambda + i0)f, g\rangle d\lambda, \tag{2.7}$$
converges for $f, g \in \mathcal{D}$. By the same method, one can show that the integral defining $K_\epsilon(t)$ converges uniformly in $\epsilon > 0$ and one has
$$\langle F(t)f, g\rangle = \lim_{R,\epsilon\to 0_+}\langle F_\epsilon(t)f, g\rangle$$

for any $f, g \in \mathcal{D}$. The bound on $\|F_\epsilon(t)\|$ shows that $|\langle F(t)f, g\rangle| \le \|f\|\|g\|$. Therefore $F(t)$ can be extended to a contraction on $\mathcal{H}$, still denoted by $F(t)$ and $\lim_{\epsilon\to 0_+} F_\epsilon(t) = F(t)$ weakly for $t > 0$.

For $f \in \mathcal{D} \cap D(H)$ with $Hf \in \mathcal{D}$ and $g \in \mathcal{D}$, one has

$$\int_{\Gamma_{\epsilon,R_0}} \frac{d}{dt}\langle e^{-itz}(H_\epsilon - z)^{-1} f, g\rangle dz = -ie^{-i\epsilon}\int_{\Gamma_{\epsilon,R_0}} \langle e^{-itz}(H_\epsilon - z)^{-1} Hf, g\rangle dz$$

converges uniformly in $\epsilon, R_0 > 0$ small. It follows that $\langle F(t)f, g\rangle$ is differentiable in $t > 0$ and

$$\langle \frac{dF(t)}{dt} f, g\rangle = \langle -iF(t)Hf, g\rangle = \langle -iHF(t)f, g\rangle.$$

By an argument of density, we deduce that for any $f \in D(H)$, $t \to F(t)f$ is weakly differentiable and

$$\frac{dF(t)}{dt} f = -iF(t)Hf = -iHF(t)f. \tag{2.8}$$

To show that $F(t) = e^{-itH}$, we prove that $F(t) \to 1$ weakly as $t \to 0_+$. For each $t \in]0, 1]$, take $R_0 = t^{-1} \ge 1$. Making a change of variables, we obtain that

$$F_\epsilon(t) = \frac{1}{2\pi i}\int_{\Gamma_{\epsilon,1}} e^{-i\zeta}\zeta^{-1}(H_\epsilon - \frac{\zeta}{t})^{-1}\frac{d\zeta}{t}, \quad t \in]0, 1].$$

Noticing that

$$\frac{1}{2\pi i}\int_{\Gamma_{\epsilon,1}} e^{-i\zeta}\zeta^{-1} d\zeta = -1$$

for every $\epsilon > 0$, one deduces

$$\langle (F_\epsilon(t) - 1)f, g\rangle = \frac{1}{2\pi i}\int_{\Gamma_{\epsilon,1}} e^{-i\zeta}\zeta^{-1}\langle (H - \frac{\zeta e^{i\epsilon}}{t})^{-1} Hf, g\rangle d\zeta \tag{2.9}$$

for $f \in D(H)$. For $\zeta \in \Gamma_{\epsilon,1}$, one has $|\zeta| \ge \frac{1}{2}$ and $|e^{-i\zeta}| \le C$ uniformly in $\epsilon > 0$ small. By the condition (2.2),

$$|\langle (H - \frac{\zeta e^{i\epsilon}}{t})^{-1} Hf, g\rangle| \le C_{f,g}(\frac{t}{|\zeta|})^\rho, \quad t \in]0, 1],$$

for any $f, g \in vD$ with $Hf \in \mathcal{D}$, uniformly in $\epsilon > 0$. It follows that

$$|\langle (F_\epsilon(t) - 1)f, g\rangle| \le C' t^\rho, \quad t \in]0, 1], \tag{2.10}$$

uniformly in ϵ. Taking the limit $\epsilon \to 0$ in the above inequality, we obtain

$$|\langle (F(t) - 1)f, g\rangle| \le C' t^\rho. \tag{2.11}$$

This proves that $\lim_{t\to 0_+}\langle (F(t) - 1)f, g\rangle = 0$ for any $g \in \mathcal{D}$ and $f \in \mathcal{D} \cap D(H)$ with $Hf \in \mathcal{D}$. Since $\|F(t) - 1\| \le 2$, an argument of density shows that $F(t)f \to f$ weakly for any $f \in \mathcal{H}$ as $t \to 0_+$.

Now for any $f \in D(H)$, let $f(t) = F(t)f$ and $u(t) = e^{-itH}f$. Then $f(t) \in D(H)$ for any $t > 0$ and one has

$$\frac{d}{ds}\langle F(s)e^{-i(t-s)H}f, g\rangle = \langle F(s)(iH - iH)e^{-i(t-s)H}f, g\rangle = 0, 0 < s \le t,$$

for any $g \in \mathcal{H}$. Integrating the above equation gives that $\langle f(t), g\rangle - \langle u(t), g\rangle = c$ for some constant c. Since both $f(t)$ and $u(t)$ converge weakly to f as $t \to 0_+$, one has $c = 0$, hence $\langle f(t), g\rangle = \langle u(t), g\rangle$ for any $g \in \mathcal{H}$. This proves that $F(t)f = e^{-itH}f$ for any $f \in D(H)$. Since $D(H)$ is dense in $\mathcal{H}$, $F(t)$ coincides with the semigroup generated by $-iH$. □

Another consequence of global limiting absorption principle is the Kato's smoothness estimate for semigroup of contractions which is useful for dissipative quantum scattering. We give below a simple proof, using the theory of selfadjoint dilation. See also [16] in some special case.

Let H be maximal dissipative on a Hilbert space $\mathcal{H}$. $-iH$ is generator of a semigroup of contractions $T(s) = e^{-isH}$, $t \geq 0$. According to the theory of Foiaş-Sz. Nagy (Masson,1967, Ch.III, § 9), $\exists$ a Hilbert space $\mathcal{G} \supset \mathcal{H}$ and a unitary group $U(t) = e^{-itG}$ on $\mathcal{G}$ such that

$$\Pi_0 U(s)|_{\mathcal{H}} = T(s),\ s \geq 0, \tag{2.12}$$

where $\Pi_0 : \mathcal{G} \to \mathcal{H}$ is the projection. G is called a *selfadjoint dilation* of H.

th2.2 **Theorem 2.2.** *Assume that there exits $A : \mathcal{H} \to \mathcal{H}$ continuous such that*

$$\sup_{\lambda\in\mathbb{R},\delta\in]0,1]} \|A(H - (\lambda + i\delta))^{-1}A^*\| \leq \gamma. \tag{2.13}$$

Then

$$\int_0^\infty (\|AT(s)f\|^2 + \|AT(s)^*f\|^2) ds \leq 2\gamma\|f\|^2, \quad f \in \mathcal{H}. \tag{2.14}$$

Proof. Let G be a selfadjoint dilation of H. Then

$$\Pi_0(G - z)^{-1}|_{\mathcal{H}} = (H - z)^{-1},\ \Pi_0(G - \overline{z})^{-1}|_{\mathcal{H}} = (H^* - \overline{z})^{-1},$$

for $\Im z > 0$. Therefore

$$\|(A\Pi_0)(G - z)^{-1}(A\Pi_0)^*\| \leq \gamma, \quad 0 < |\Im z| \leq 1.$$

By Kato's smoothness estimate for selfadjoint operators,

$$\int_{-\infty}^\infty \|A\Pi_0 U(s)g\|^2\, ds \leq C\|g\|^2, \quad g \in \mathcal{G},$$

with

$$C = \sup_{0<\Im z\leq 1} \|(A\Pi_0)[(G - z)^{-1} - (G - \overline{z})^{-1})(A\Pi_0)^*\| \leq 2\gamma.$$

For $g = f \in \mathcal{H}$, one has

$$\int_0^\infty (\|AT(s)f\|^2 + \|AT(s)^*f\|^2)\, ds \leq 2\gamma\|f\|^2, \quad f \in \mathcal{H}.$$

□

3. Dissipative Schrödinger operators

Let $R(z) = (H - z)^{-1}$, $z \notin \sigma(H)$. Let $R_j(z) = (H_j - z)^{-1}$. Denote $L^{2,s} = L^2(\mathbb{R}^n; \langle x\rangle^s dx)$ and $\|f\|_s = \|f\|_{L^{2,s}}$. It is well know that if $n \geq 3$, the limit

$$F_0 = \lim_{z\to 0, z\notin\mathbb{R}_+} R_0(z) : L^{2,s} \to L^{2,-s}$$

exists if $s > 1$.

prop1 **Proposition 3.1.** *Assume that $n \geq 3$ and $\rho_0 > 2$. Then one has*

(a). $1 + F_0 V$ is invertible on $L^{2,-s}$ for any $s \in]1, \rho_0/2[$ and there exists $c_0 > 0$ such that the limit

$$R(\lambda + i0) = \lim_{\epsilon\to 0_+} R(\lambda + i\epsilon) : L^{2,s} \to L^{2,-s}$$

exists for $s > 1$ and $\lambda \in [-c_0, c_0]$.

(b). Zero is not an accumulating point of the eigenvalues of H and there exists $\delta_0 > 0$ such that

$$\sigma(H) \subset \{z; -\pi + \delta_0 \leq \arg z \leq 0\}. \tag{3.1}$$

This result is proved by X.P. Wang 2009.

th1 **Theorem 3.2.** *Assume that $n \geq 3$ and $\rho_0 > 2$. Then*

(a). For any $s > 1$

$$\sup_{\epsilon\in]0,1],\lambda\in\mathbb{R}} \langle\lambda\rangle^{\frac{1}{2}} \|\langle x\rangle^{-s} R(\lambda + i\epsilon)\langle x\rangle^{-s}\| < \infty \tag{3.2}$$ global-re

The limit

$$R(\lambda + i0) = \lim_{\epsilon\to 0_+} R(\lambda + i\epsilon) : \quad L^{2,s} \to L^{2,-s}$$

exists for $s > 1$ and is continuous on $\mathbb{R}$.

(b) Let $k \in \mathbb{N}$. Assume that $|(x\cdot\nabla)^j V(x)| \leq C\langle x\rangle^{-\rho_0}$, $j = 0, 1, \cdots, k$, $\rho_0 > 2$. Then for $j = 0, 1, \cdots, k + 1$ and $s > j + \frac{1}{2}$, one has

$$\|\langle x\rangle^{-s} \frac{d^j}{d\lambda^j} R(\lambda + i0)\langle x\rangle^{-s}\| \leq C_s \langle\lambda\rangle^{-\frac{j+1}{2}}, \tag{3.3}$$

for $\lambda > 1$.

Ideas. For $\lambda > 0$, one uses the equation $R(z) = (1 + R_0(z)V)^{-1} R_0(z)$ by the argument of S. Agmon (1975) ($\rho_0 > 1$). For more general situations (an abstract Mourre's theory for dissipative operators), see J. Royer (Commun. in PDE, to appear). For λ near 0,
prop1 we apply Proposition . $n \geq 3$ and $\rho_0 > 2$ are needed.

4. Optimal time-decay rate

As an application of the representation of the semigroup, we can show the following dispersive estimate for dissipative Schrödinger operators.

dispersive **Theorem 4.1.** *Assume $n = 3$ and $|V(x)| + |x\cdot\nabla V(x)| \leq C\langle x\rangle^{-\rho_0}$, $\rho_0 > 2$. Then one has*

$$\|e^{-itH} f\|_{L^\infty} \leq C t^{-\frac{3}{2}} \|f\|_{L^1}, \quad \forall\, f \in L^1(\mathbb{R}^3), t > 0. \tag{4.1}$$ dispative-

To prove the Theorem, it suffices to prove

$$|\langle U(t)u, v\rangle| \le C|t|^{-3/2}\|u\|_{L^1}\|v\|_{L^1}, \quad u, v \in C_0^\infty.$$

Distinguish two regimes: $\lambda \in [-R, R]$ and $|\lambda| > R$. Consider only the case $\lambda \ge 0$. By the change of variable $\lambda \to \lambda^2$ and an integration by parts, we are led to prove

$$|\int e^{-it\lambda^2}\rho(\lambda)\langle G'(\lambda)u, v\rangle \, d\lambda| \le C|t|^{-1/2}\|u\|_1\|v\|_1. \tag{4.2}$$ est1

Here ρ is an appropriate cut-off and $G(\lambda) = r(\lambda^2)$ which can be written as $G(\lambda) = G_0(\lambda)(1 + VG_0(\lambda))^{-1}$.

One can calculate

$$G'(\lambda) = (1 - G_0(\lambda)V)G_0'(\lambda)(1 + VG_0(\lambda))^{-1} \tag{4.3}$$ G

and $G_0'(\lambda)$ is the operator with integral kernel: $i\frac{e^{i\lambda|x-y|}}{4\pi}$.

Take ρ with support in $]\lambda_0 - \delta, \lambda_0 + \delta[$, $\delta > 0$ to be adjusted. If one replaces λ by λ_0 in $(1 - G_0(\lambda)V)$ and $(1 + VG_0(\lambda))^{-1}$ in (4.3), one sees that

$$|\int_{\mathbb{R}^3_x}\int_{\mathbb{R}^3_y}\int_R e^{-it\lambda^2 + i\lambda|x-y|}\rho(\lambda)\tilde{u}(x)\overline{\tilde{v}(y)}d\lambda dx dy| \le C|t|^{-1/2}\|u\|_1\|v\|_1.$$

Note that $VG_0(\lambda) : L^1 \to L^1$ is Hölder-continuous in λ. For $\lambda \in$ *supp* ρ, one expands

$$(1 + VG_0(\lambda))^{-1} = \sum_{k=0}^{\infty}(-1)^k(S_0D(\lambda))^k S_0$$

with $S_0 = 1 + VG_0(\lambda_0), D(\lambda) = V(G_0(\lambda) - G_0(\lambda_0))$. The integral kernel of $D(\lambda)$ is

$$V(x)\frac{e^{i\lambda|x-y|} - e^{i\lambda_0|x-y|}}{4\pi|x-y|}.$$

One can prove that

$$\int_{\mathbb{R}} \|\mathcal{F}_{\lambda\to\tau}\rho(\lambda)(S_0D(\lambda))^k S_0 u\|_1 d\tau \le (C_V\delta^\epsilon)^k\|u\|_1.$$

It follows that

$$|\int_{\mathbb{R}}\langle G_0'(\lambda)e^{-it\lambda^2}\rho(\lambda)(S_0D(\lambda))^k S_0 u, A(\lambda)v\rangle d\lambda| \le |t|^{-1/2}(C_V\delta^\epsilon)^k\|u\|_1\|v\|_1.$$

Here $A(\lambda) = 1 - VG_0(\lambda_0)$ or $D(\lambda)$.

Taking δ s.t. $C_V\delta^\epsilon < 1$ and summing up in k, we obtain the desired estimate (4.2) for each fixed energy.

Little modification is needed when $\lambda_0 = 0$.

The high energy estimate in the limiting absorption principle implies

$$\|(VG_0(\lambda))^k u\|_1 \le C^k\lambda^{-(k-2)}\|u\|_1$$

We also need the following

Lemma 4.2. *Let $f_k : \mathbb{R} \to \mathbb{R}$ be a sequence of continuous functions such that*

$$|f_k(\lambda)| \le C^k \langle \lambda \rangle^{-k+2}, \quad \int_{\mathbb{R}} \langle \tau \rangle^{\epsilon_0} |\hat{f}_k(\tau)| d\tau \le C^k.$$

Then for χ a cut-off for the interval $[R, \infty[$, one has

$$\int |\widehat{\chi f_k}(\tau)| d\tau \le C_1^k \langle R \rangle^{-\frac{(k-2)\epsilon_0}{2(1+\epsilon_0)}}.$$

One uses the series for λ large

$$(1 + VG_0(\lambda))^{-1} = \sum_{k=0}^{\infty} (-1)^k (VG_0(\lambda))^k$$

By the decay condition on V, one has

$$\int \langle \tau \rangle^{\epsilon_0} \|\mathcal{F}_{\lambda \to \tau} \rho(\lambda)(VG_0(\lambda))^k u\|_1 d\tau \le C^k \|u\|_1,$$

uniformly in R. Here ρ is a cut-off for $[R, \infty[$. The Lemma gives

$$\int_{\mathbb{R}} \|\mathcal{F}_{\lambda \to \tau} \rho (VG_0(\lambda))^k u\|_1 d\tau \le C^k R^{-\frac{(k-2)\epsilon_0}{2(1+\epsilon_0)}} \|u\|_1.$$

For $R > 1$ large enough,

$$C^k R^{-\frac{(k-2)\epsilon_0}{2(1+\epsilon_0)}} < \epsilon^k, \epsilon < 1.$$

One can deduce that

$$|\int e^{-it\lambda^2} \rho(\lambda) \langle G_0'(\lambda)(VG_0(\lambda))^k u, A(\lambda) v \rangle d\lambda| \le |t|^{-1/2} C \epsilon^k \|u\|_1 \|v\|_1.$$

Taking the summation in k, we obtain the desired high energy estimate. □

This result is to compare with the dispersive estimate for selfadjoint operator H_1 ($V_2 = 0$). Assume that $n = 3$ and $|V_1(x)| \le C \langle x \rangle^{-\rho_0}$, $\rho_0 > 2$ and that 0 is not an eigenvalue nor a resonance of H_1. Then one has

$$\|e^{-itH_1} P_{ac} f\|_{L^\infty} \le C t^{-\frac{3}{2}} \|f\|_{L^1}, \quad \forall\, f \in L^1(\mathbb{R}^3), t \ne 0, \tag{4.4}$$

where P_{ac} is the projection onto the absolutely continuous spectral subspace of H_1.

As a consequence of Theorem 4.1, one has

Corollary 4.3. *Under the condition of Theorem 4.1, one has for any $s \in [0, 3/2]$ and $s' > s$,*

$$\|\langle x \rangle^{-s'} e^{-itH} \langle x \rangle^{-s'}\|_{\mathcal{L}(L^2)} \le C \langle t \rangle^{-s}, \quad t > 0. \tag{4.5}$$

Proof. For $s = 3/2$ and $s' > 3/2$, it follows from Theorem 4.1 that the above estimate holds. The general case follows from an argument of interpolation. □

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